PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 4

ONE-DIMENSIONAL UNSTEADY GAS

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Consider an unsteady 1-dimensional ideal gas flow. We assume that this flow is spatially continuous and thermally isolated, hence, it is **isentropic**.

The governing equations are

Mass conservation: $\frac{\partial}{\partial t}\rho + u\frac{\partial}{\partial x}\rho + \rho\frac{\partial}{\partial x}u = 0$

Equation of motion (Euler): $\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + \frac{1}{\rho}\frac{\partial}{\partial x}p = 0$

Since the flow is assumed isentropic we also have $\left.\frac{dp}{d\rho}\right|_{s=const} = a^2$ and

$$\frac{\partial}{\partial x}p = \left.\frac{dp}{d\rho}\right|_{s=const}\frac{\partial}{\partial x}\rho = a^2\frac{\partial}{\partial x}\rho$$

We introduce the function of gas density by the formula

$$\hat{P}(\rho) = \int^{\rho}\frac{a}{\rho}d\rho \;\;\Rightarrow\;\; \hat{P}'(\rho) = \frac{a}{\rho}$$

The corresponding function of time and spatial coordinate can be defined

$$P(t,x) = \hat{P}[\rho(t,x)]$$

We will need derivatives:

$$\frac{\partial}{\partial t} P(t,x) = \hat{P}'[\rho(t,x)]\frac{\partial}{\partial t}\rho(t,x) = \frac{a}{\rho}\frac{\partial}{\partial t}\rho(t,x)$$

$$\frac{\partial}{\partial x} P(t,x) = \hat{P}'[\rho(t,x)]\frac{\partial}{\partial x}\rho(t,x) = \frac{a}{\rho}\frac{\partial}{\partial x}\rho(t,x)$$

Next, we have

$$\frac{\partial}{\partial x} p = a^2 \frac{\partial}{\partial x}\rho \quad \Rightarrow \quad \frac{\partial}{\partial x}\rho = \frac{1}{a^2}\frac{\partial}{\partial x} p$$

and

$$\frac{\partial}{\partial x} P = \frac{a}{\rho}\frac{\partial}{\partial x}\rho = \frac{1}{\rho a}\frac{\partial}{\partial x} p \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial x} p = a\frac{\partial}{\partial x} P$$

Using the above relation, the equation of motion can be written as

$$\frac{\partial}{\partial t} u + u\frac{\partial}{\partial x} u + a\frac{\partial}{\partial x} P = 0 \text{(*)}$$

The mass conservation equation can be multiplied by a/ρ

$$\underbrace{\frac{a}{\rho}\frac{\partial}{\partial t}\rho}_{\frac{\partial}{\partial t}P} + u\underbrace{\frac{a}{\rho}\frac{\partial}{\partial x}\rho}_{\frac{\partial}{\partial x}P} + \frac{a}{\rho}\rho\frac{\partial}{\partial x}u = 0$$

which leads to its equivalent form

$$\frac{\partial}{\partial t}P + u\frac{\partial}{\partial x}P + a\frac{\partial}{\partial x}u = 0 \qquad (**)$$

Addition and subtraction of the equations (*) and (**) yield the system of equations

$$\frac{\partial}{\partial t}(u+P) + (u+a)\frac{\partial}{\partial x}(u+P) = 0$$
$$\frac{\partial}{\partial t}(u-P) + (u-a)\frac{\partial}{\partial x}(u-P) = 0$$

What is a physical meaning of these equations?

Consider the function $f = f(t,x)$ and the line $l : x = X(t)$. One can define the function

$$F(t) := f\Big|_l (t,x) = f[t, X(t)]$$

Let's calculate the (ordinary) derivative of the function F. We have

$$F'(t) = \tfrac{\partial}{\partial t} f[t, X(t)] + \tfrac{\partial}{\partial x} f[t, X(t)]\, X'(t)$$

We see that if $\tfrac{\partial}{\partial t} f[t, X(t)] + \tfrac{\partial}{\partial x} f[t, X(t)]\, X'(t) = 0$ then the derivative $F'(t)$ vanishes identically meaning that the function $f = f(t,x)$ is constant (or invariant) along the line l!

Looking at the obtained system of the differential equations we conclude that:

$r = u + P$ is constant along each line C_+ such that $X'(t) = (u+a)[t, X(t)]$
$s = u - P$ is constant along each line C_- such that $X'(t) = (u-a)[t, X(t)]$

It remains to calculate the explicit form of the function P …

Since the flow is isentropic we have the relation $p = K\rho^{\kappa}$, $\kappa = \dfrac{c_p}{c_v}$.

Thus
$$a^2 = \kappa\frac{p}{\rho} = \kappa K\rho^{\kappa-1} \Rightarrow a = \sqrt{\kappa K}\,\rho^{\frac{\kappa-1}{2}}$$

and
$$\hat{P} = \int^{\rho}\frac{a}{\rho}d\rho = \int^{\rho}\sqrt{\kappa K}\,\rho^{\frac{\kappa-1}{2}-1}d\rho = \tfrac{2}{\kappa-1}\underbrace{\sqrt{\kappa K}\,\rho^{\frac{\kappa-1}{2}}}_{a} = \frac{2a}{\kappa-1}$$

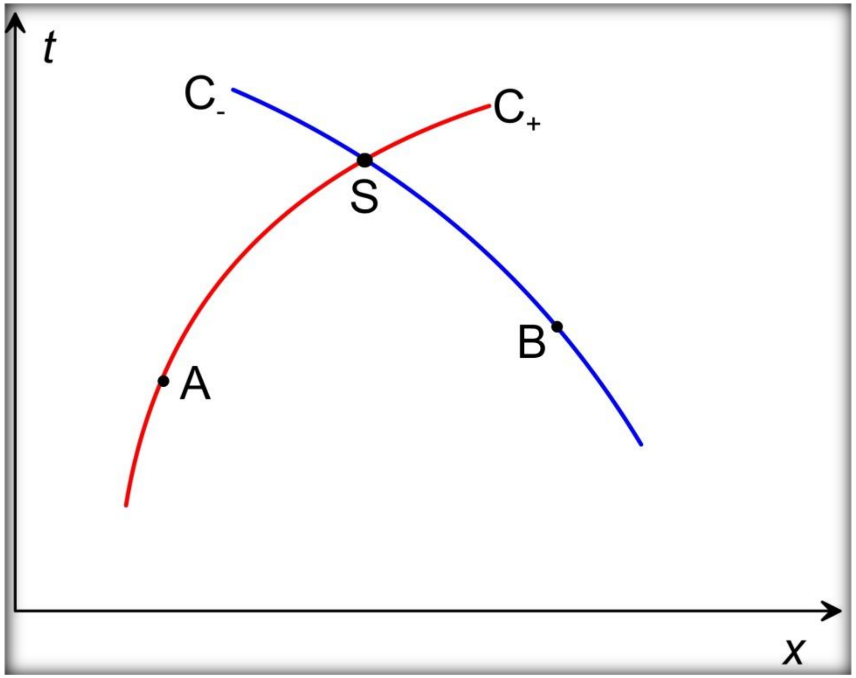
Hence, the explicit formulae for the conserved quantities are
$$r = u + \tfrac{2}{\kappa-1}a \quad , \quad s = u - \tfrac{2}{\kappa-1}a$$

Summarizing - in a 1D unsteady flow of the Clapeyron gas:

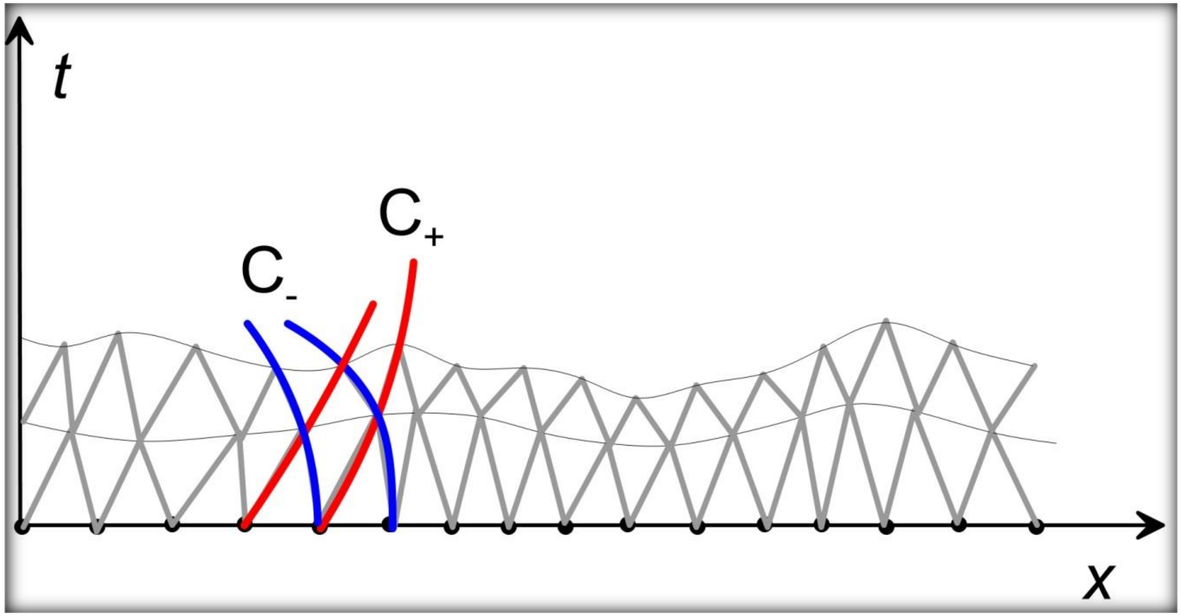
- The quantity $r = u + \tfrac{2}{\kappa-1}a$ (the 1st Riemann invariant) is constant along each line such that $X'(t) = (u+a)[t, X(t)]$. We call these lines the C_+ characteristics.
- The quantity $s = u - \tfrac{2}{\kappa-1}a$ (the 2nd Riemann invariant) is constant along each line such that $X'(t) = (u-a)[t, X(t)]$. We call these lines the C_- characteristics.

What is the gain of knowing the shape of characteristic lines? Assume we know that the characteristics, one C_+ and the other C_-, intersecting at a given point S (see figure) go through the points A and B , respectively. We assume also that both velocity and the speed of sound is known for the points A and B. Then, we can easily determine the velocity and the speed of sound in the point S:

$$\begin{cases} u_S + \frac{2}{k-1} a_S = u_A + \frac{2}{k-1} a_A \\ u_S - \frac{2}{k-1} a_S = u_{\boldsymbol{B}} + \frac{2}{k-1} a_B \end{cases} \Rightarrow \begin{cases} u_S = \frac{1}{2}(u_A + u_B) + \frac{1}{k-1}(a_A - a_B) \\ a_S = \frac{1}{2}(a_A + a_B) + \frac{k-1}{4}(u_A - u_B) \end{cases}$$



- Note that the above solution is possible only when the shape of the characteristic lines is a priori known, which – in general – is not the case, because the slope of these lines depends on unknown velocity and the speed of sound!
- There exist a few special cases when the flow problem can be solved in the closed form, though. These cases concern the flows with simple waves.
- Conceptually, it is possible to device an approximate method based on the use of characteristics – see next figure.



For $t = 0$ both velocity and the speed of sound are known at every point of the flow domain (the initial conditions). We divide the spatial domain into short segments. Using the initial conditions, the initial slopes of the characteristic lines of both kind can be found. Then, these lines are advanced forward in time (as straight lines) until first intersections. Note that – in general – the intersections do not correspond to te same time instant. The flow parameters at the intersections can be computed using Riemann invariants from the “root” points.

This procedure can be continued to produce next layer of intersection. Having more layers, the shape of the characteristics can be approximated by higher-order extrapolation method. In order to avoid to large “deformations” of the “intersection grid”, the re-meshing procedure should be used (interpolation of the flow parameter to some $t = const$ layer).

Theorem: If in some region in the *(x,t)* plane one of the Riemann invariant is constant, then the characteristics corresponding to the other invariant are the straight lines.

Proof:

Assume, without scarifying generality, that in some region the first Riemann invariant is constant. It means that at every point in this region

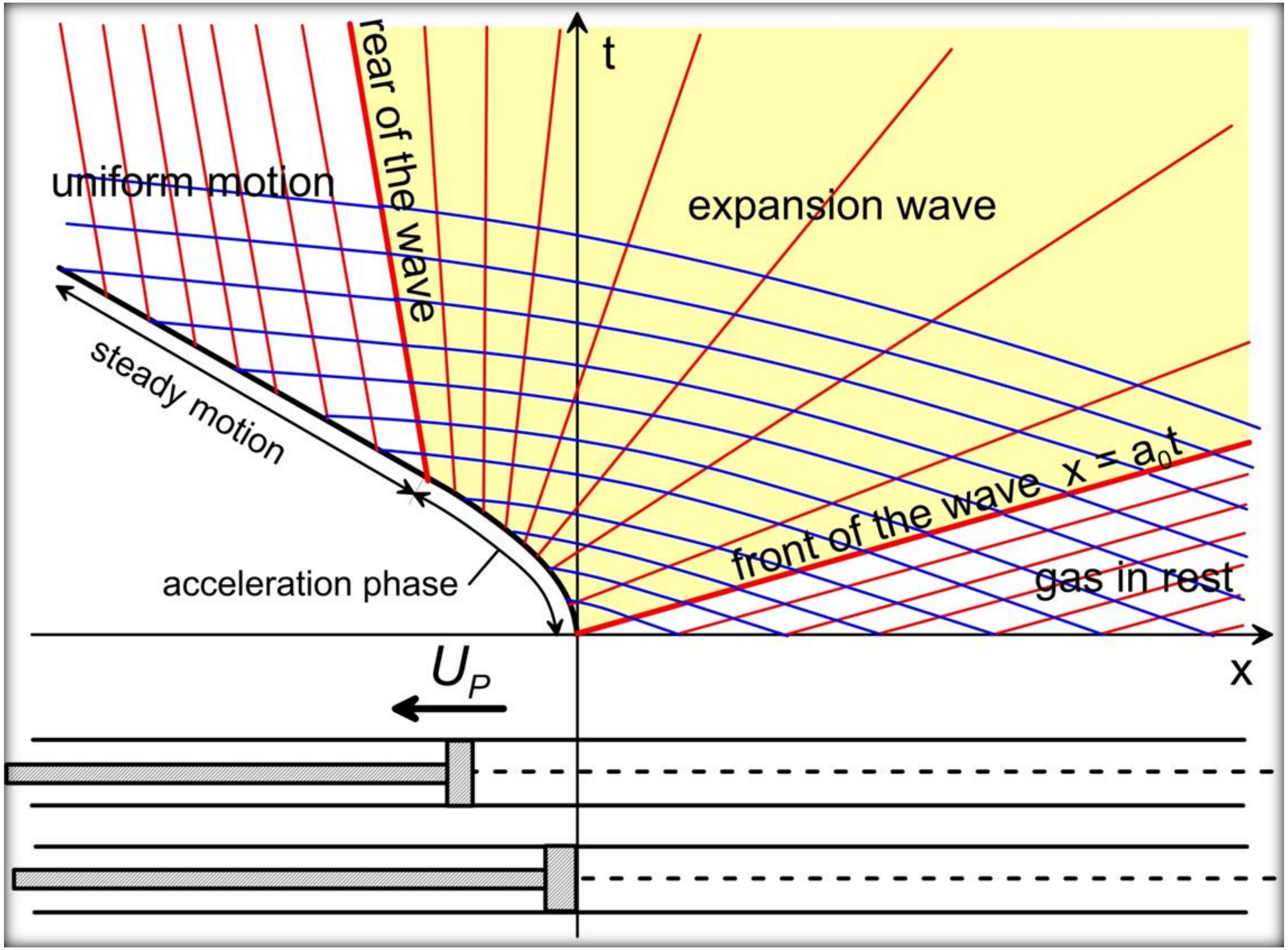
$$r = u + \frac{2}{\kappa - 1} a = const$$

Consider any characteristics C_{-} traversing the region of constant r. Then, at each point belonging to this characteristics both Riemann invariant are constant, hence, the velocity u and the speed of sound a are also constant. This, in turn, means that the slope of the characteristics C_{-}, which is equal to $u - a$ is constant, hence, this characteristics is the straight line.

Remarks:

- Any *(x,t)*-region such that one of the Riemann invariants is constant is called the **simple wave.**
- If in a certain *(x,t)*-region both Riemann invariants are constant, then both velocity and the speed of sound are constant in this region. In other words, such region corresponds to the uniform steady flow.

Example 1: Expansion wave in a pipe.



C_+ - red lines, C_- - blue lines

Calculation of the parameter in the region near the piston (behind the expansion wave)

Note that the second Riemann invariant s is globally constant and equal

$$s = \left(u - \tfrac{2}{k-1} a\right)\Big|_{t=0} = -\tfrac{2}{k-1} a_0$$

Then, we can write

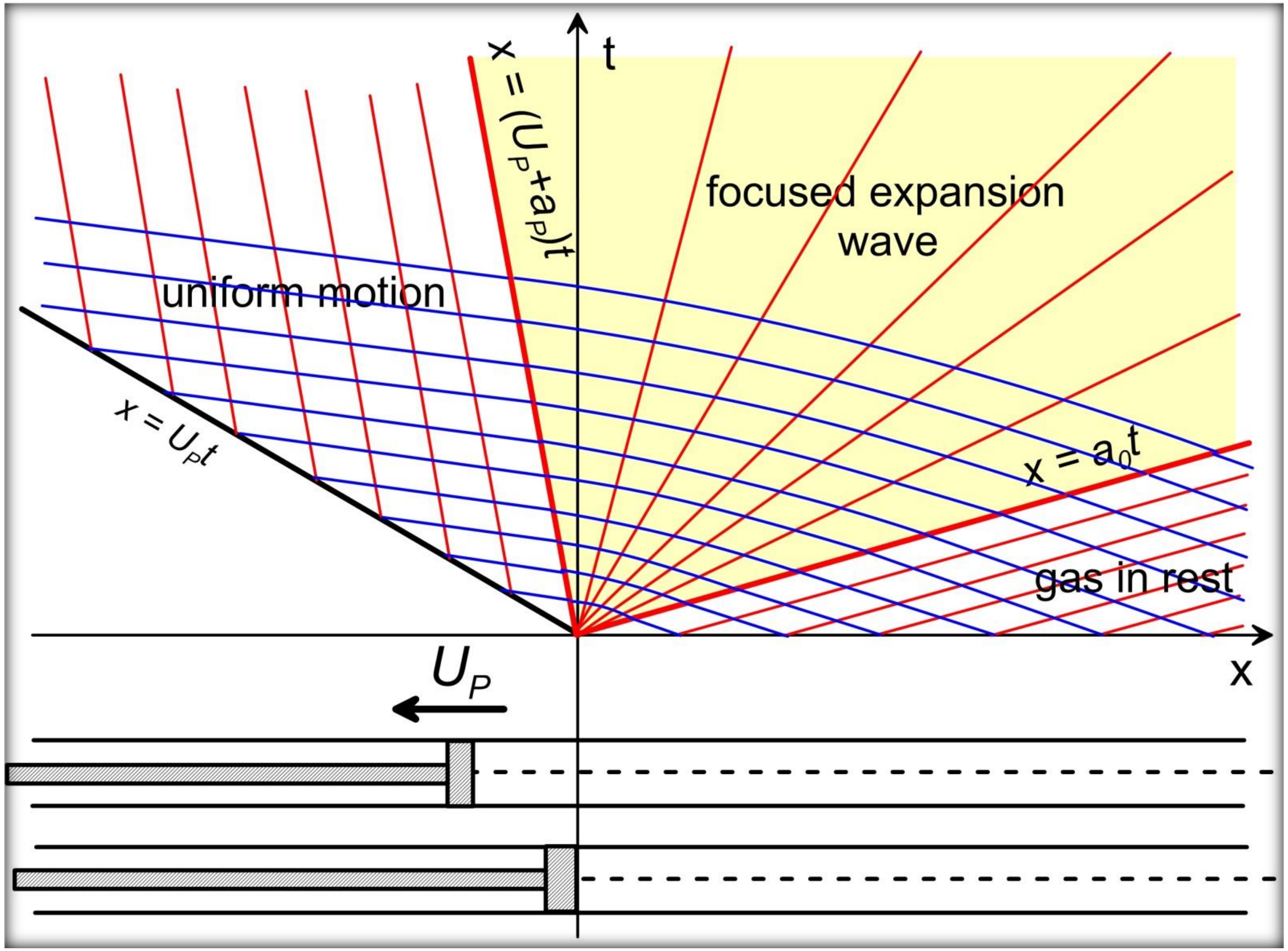
$$\left(u - \tfrac{2}{k-1} a\right)\Big|_{piston} \equiv U_P - \tfrac{2}{k-1} a_P = -\tfrac{2}{k-1} a_0$$

$$\Downarrow$$

$$a_P = a_0 + \tfrac{k-1}{2} U_P$$

Note that the sign of U_P is negative, hence, $a_P < a_0$.

When the acceleration phase shrinks to the point in time, the expansion wave take the focused form …



The point $(x,t) = (0,0)$ is a singular point – the gas goes through all intermediate states in no time.

For the focused wave we can solve the following problem: **calculate the flow velocity and the speed of sound at the duct section $x = X$ and time $t = T$.**

The point $(x,t) = (X,T)$ can be located:

- In front of the wave if $\dfrac{X}{T} > a_0$
- Inside the wave region if $U_P + a_P \le \dfrac{X}{T} \le a_0$
- Between the piston and the rear of the wave if $U_P \le \dfrac{X}{T} < U_P + a_P$
- Outside the flow domain if $\dfrac{X}{T} < U_P$

The second option is the most interesting. In order to evaluate gas parameters we note that:

- $u_X - \frac{2}{k-1} a_X = -\frac{2}{k-1} a_0$ - the second Riemann invariant is global
- $u_X + a_X = \dfrac{X}{T}$ - the (constant!) slope of the C_+ characteristics passing through the (X,T) can be expressed in two ways.

The solution is …

$$u_X = \frac{2}{k+1}\left(\frac{X}{T} - a_0\right) \quad , \quad a_X = \frac{k-1}{k+1}\frac{X}{T} + \frac{2}{k+1}a_0$$

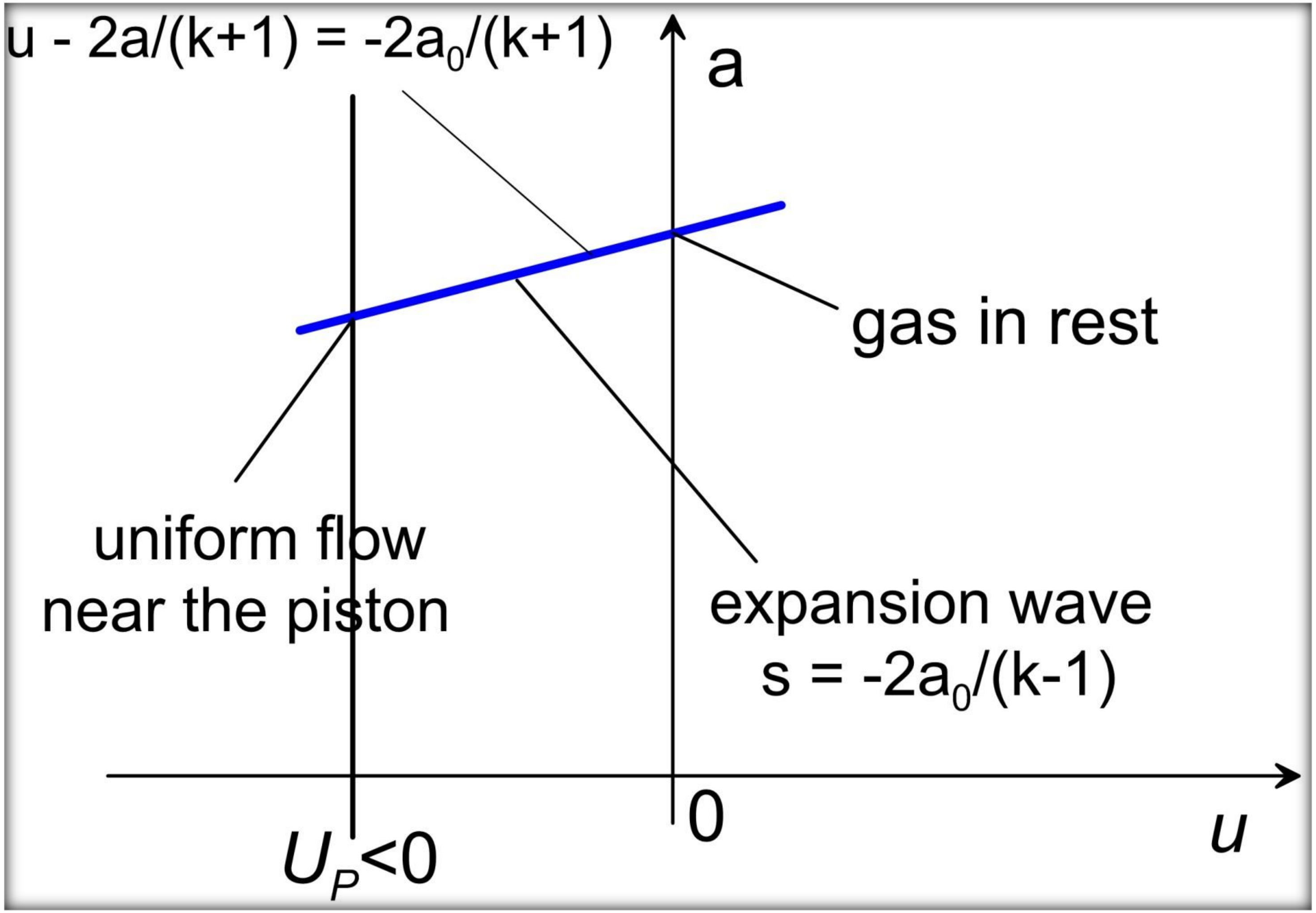
Finally, let us remind that the speed of the gas at the piston's face is equal

$$a_P = a_0 + \frac{k-1}{2}U_P \quad (U_P < 0)$$

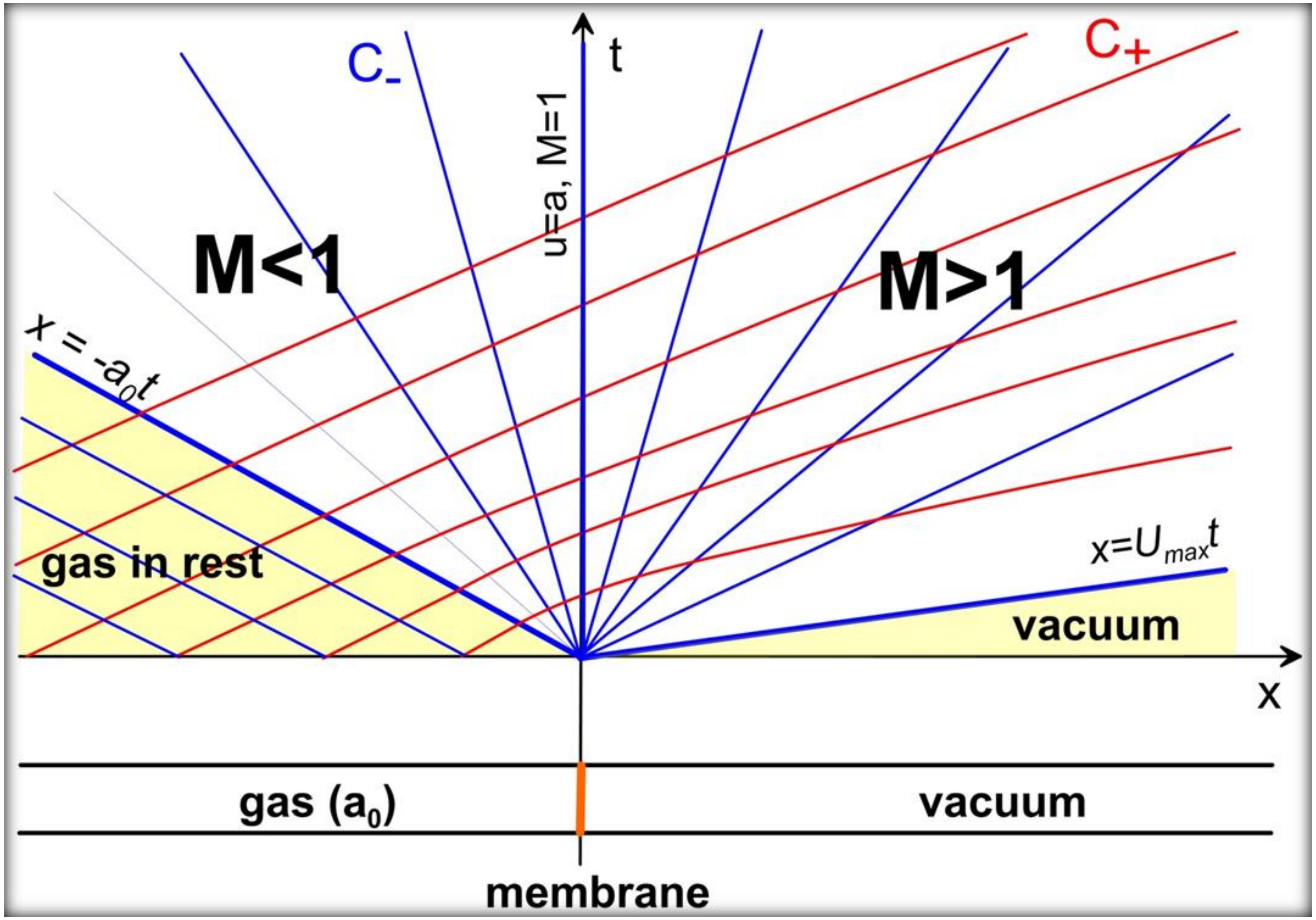
It follows that if $\left|U\right|_P > U_{\max} = \frac{2}{k-1}a_0$ then $a_P < 0$ which is physical nonsense. We conclude that if the velocity of the piston exceeds the value of $U_{\max}$ then the gas cannot catch up with the piston and the vacuum mast appear between the piston and the front of the gas stream.

Note also that **the maximal gas velocity in this (unsteady) motion is larger than the maximal velocity of gas in the steady adiabatic motion**, where $U_{\max}^{steady} = \sqrt{\frac{2}{k-1}}a_0$

Representation of the expansion wave in the (u, a) plane (the **hodograph plane**).



Example 2: Expansion wave due to the initial pressure jump



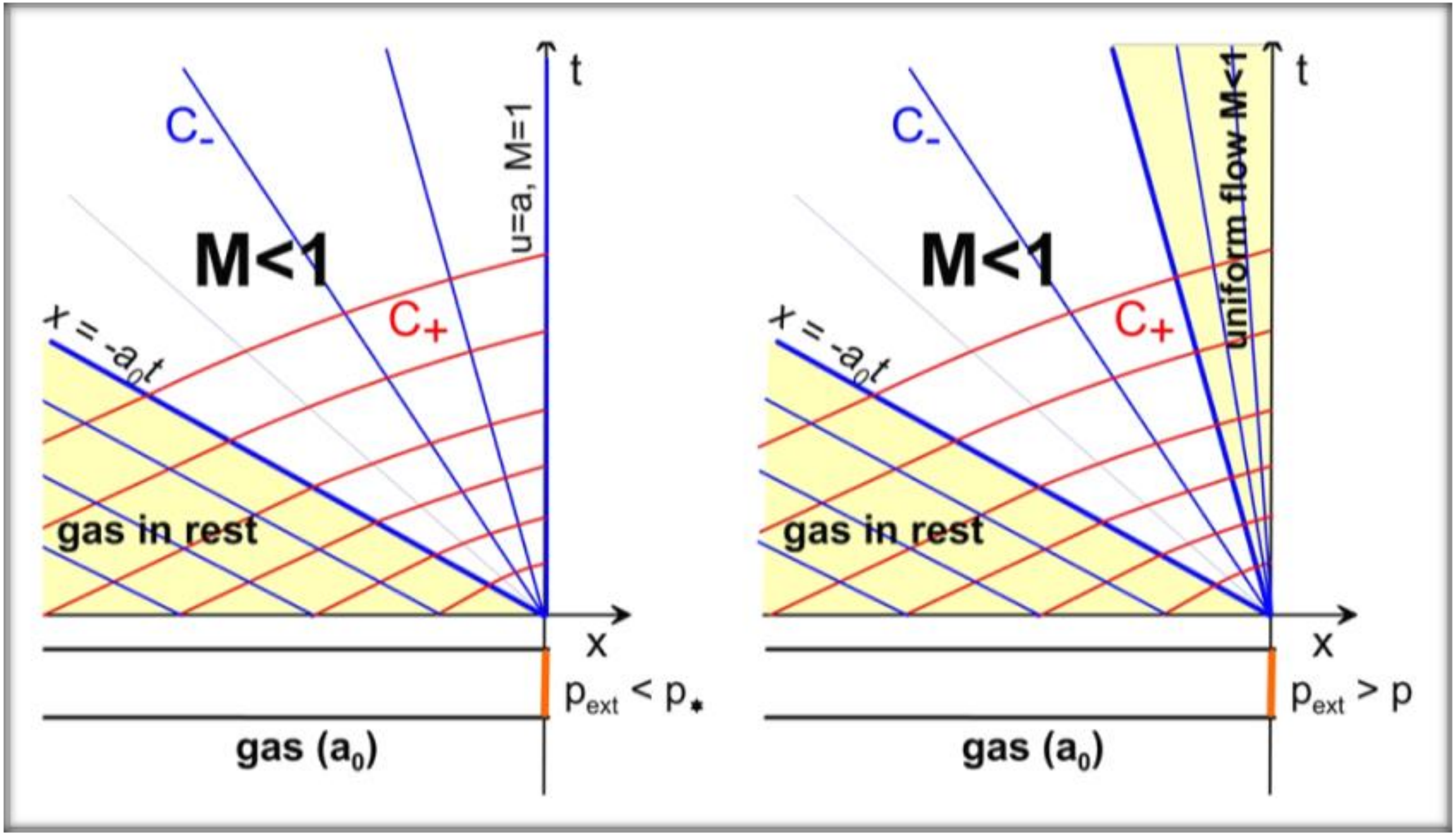
Riemann invariant r is global. The gas parameters at the given point $(x,t)=(X,T)$ can be determined from the following system of equations:

$$\begin{cases} u-a=\frac{X}{T} \\ u+\frac{2}{k-1}a=\frac{2}{k-1}a_0 \end{cases}$$

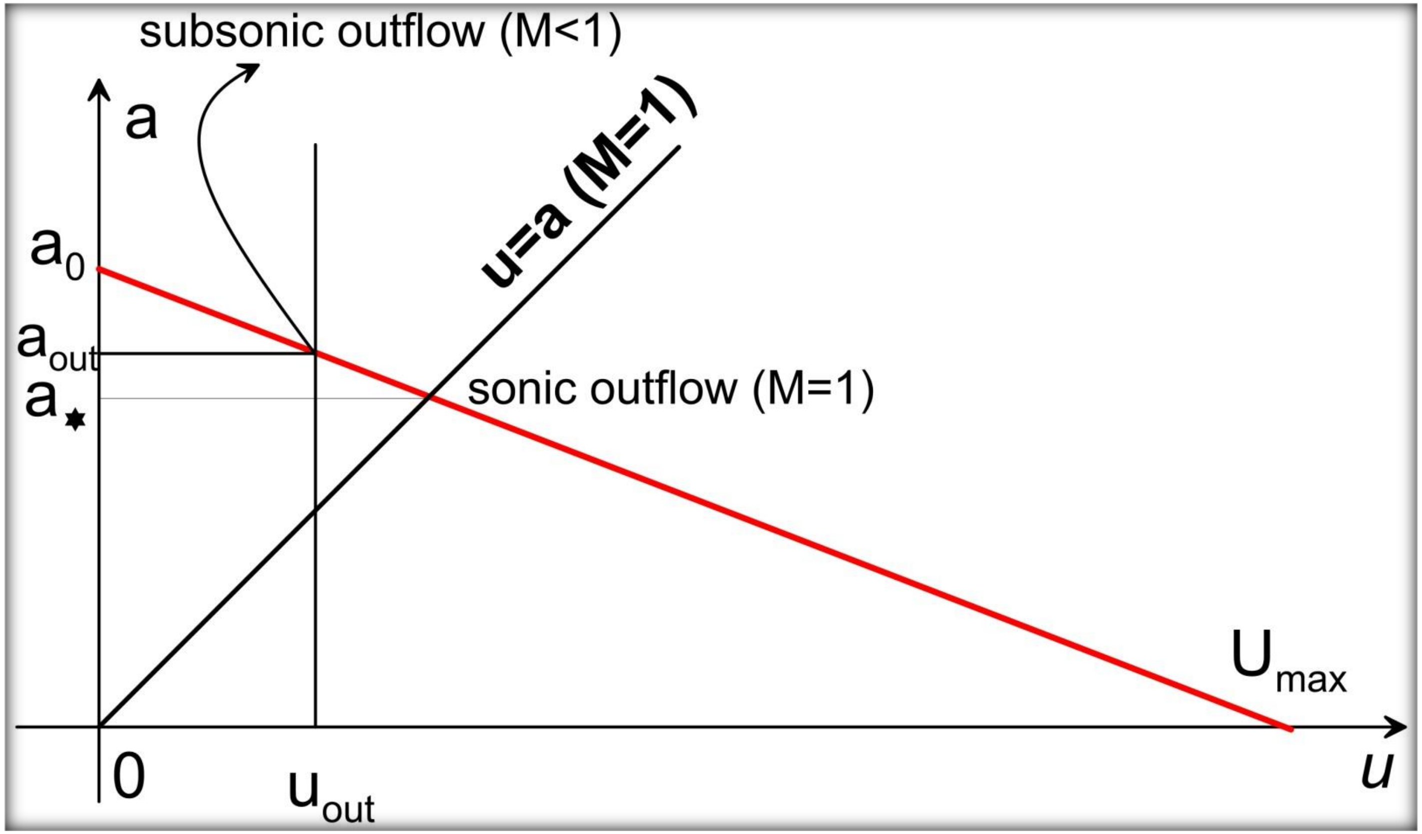
This solution is

$$\begin{cases} u=\frac{2}{k+1}\left(a_0+\frac{X}{T}\right) \\ a=\frac{2}{k+1}a_0-\frac{k-1}{k+1}\frac{X}{T} \end{cases}$$

Modification: outflow from the semi-infinite pipe.



In the hodograph plane …



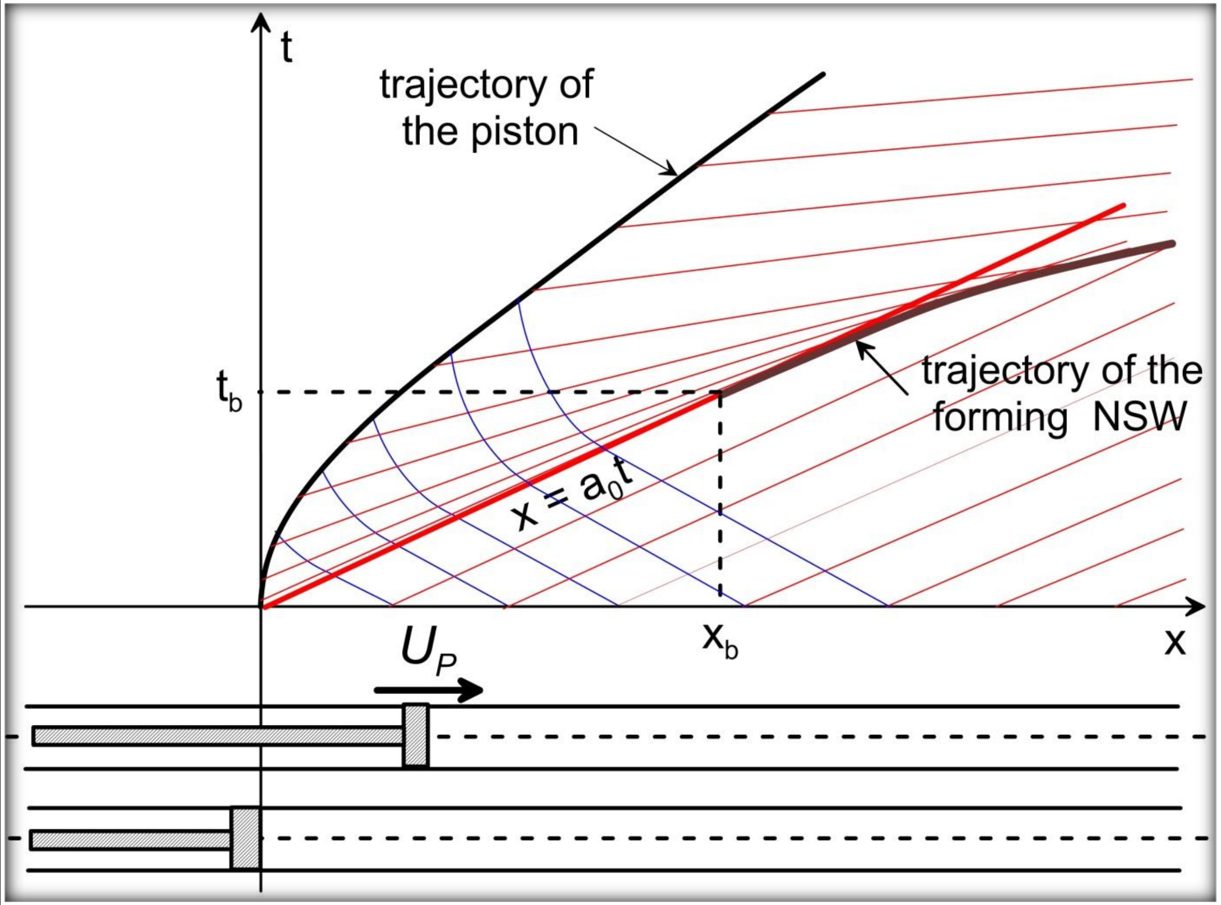
If $p_{ext} > p_*$, the outflow is subsonic and $p_{out} = p_{ext}$. From the isentropic relation (derive !):

$$a_{out} = a_0 (p_{ext}/p_0)^{(k-1)/2k}$$

The velocity u_{out} stems from the first Riemann invariant:

$$u_{out} + \frac{2}{k-1} a_{out} = \frac{2}{k-1} a_0 \quad \Rightarrow \quad u_{out} = \frac{2}{k-1} a_0 [1 - (p_{ext}/p_0)^{(k-1)/2k}]$$

Example 3: Compression wave generated by a moving piston – emergence of a discontinuity.



Trajectory of the piston: $x = X_P(t)$

In the region of continuous motion:

$$s \equiv u - \frac{2}{k-1} a = const$$

Hence

$$X'_P(t) - \frac{2}{k-1} a_P = -\frac{2}{k-1} a_0$$

$$\Downarrow$$

$$a_P = \frac{k-1}{2} X'_P + a_0$$

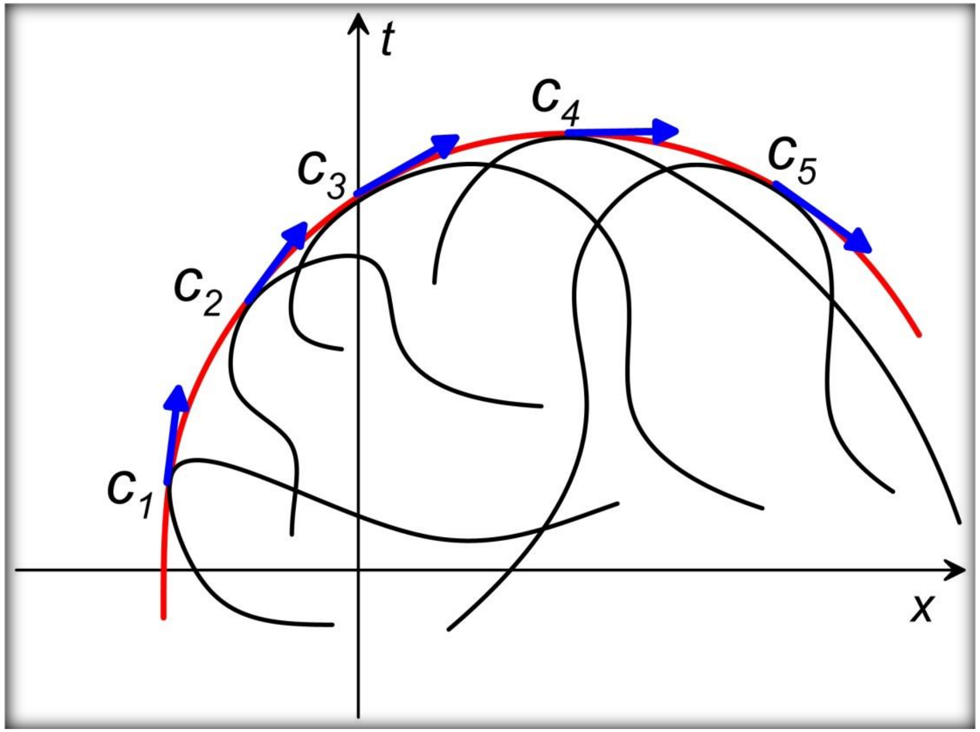
The C_+ line emerging from the piston's trajectory point $(T, X_p(T))$ is

$$x - X_P(T) = [\underbrace{a_0 + \frac{k+1}{2} X'_P(T)}_{X'_P(T) + a_P(T)}](t - T)$$

Concept of an envelope line

Assume that a family of lines in the plane is defined by an implicit formula $\varphi(c,x,t)=0$, where $c \in C$ is the parameter identifying each line in the family. The envelope of this family is the line with the parametric description $t = T_e(c),\ x = X_e(c)$, such that for each $c \in C$:

- Each point in the envelope belong to the line from the family "labeled" with this c , i.e., $\varphi[c, X_e(c), T_e(c)] = 0$
- The envelope is tangent to this line at their common point (see figure).



Tangency condition:

$$\Phi(c) \equiv \varphi[c, X_e(c), T_e(c)] = 0$$

$$\Downarrow$$

$$\Phi'(c) = \tfrac{\partial}{\partial c}\varphi[..] + \underbrace{\tfrac{\partial}{\partial x}\varphi[..]\cdot X_e'(c) + \tfrac{\partial}{\partial t}\varphi[..]\cdot T_e'(c)}_{=0!-\textit{tangency condition}}$$

$$\Downarrow$$

$$\tfrac{\partial}{\partial c}\varphi[c, X_e(c), T_e(c)] = 0$$

In our problem, $c \equiv T$ and

$$\varphi(T,x,t) = x - X_P(T) - [a_0 + \tfrac{k+1}{2} X'_P(T)](t-T) = 0$$

$$\tfrac{\partial}{\partial T}\varphi(T,x,t) = -X'_P(T) - \tfrac{k+1}{2} X''_P(T)(t-T) + [a_0 + \tfrac{k+1}{2} X'_P(T)] =$$
$$= a_0 + \tfrac{k-1}{2} X'_P(T) - \tfrac{k+1}{2} X''_P(T)(t-T) = 0$$

Hence, the parametric equations for the envelope line are:

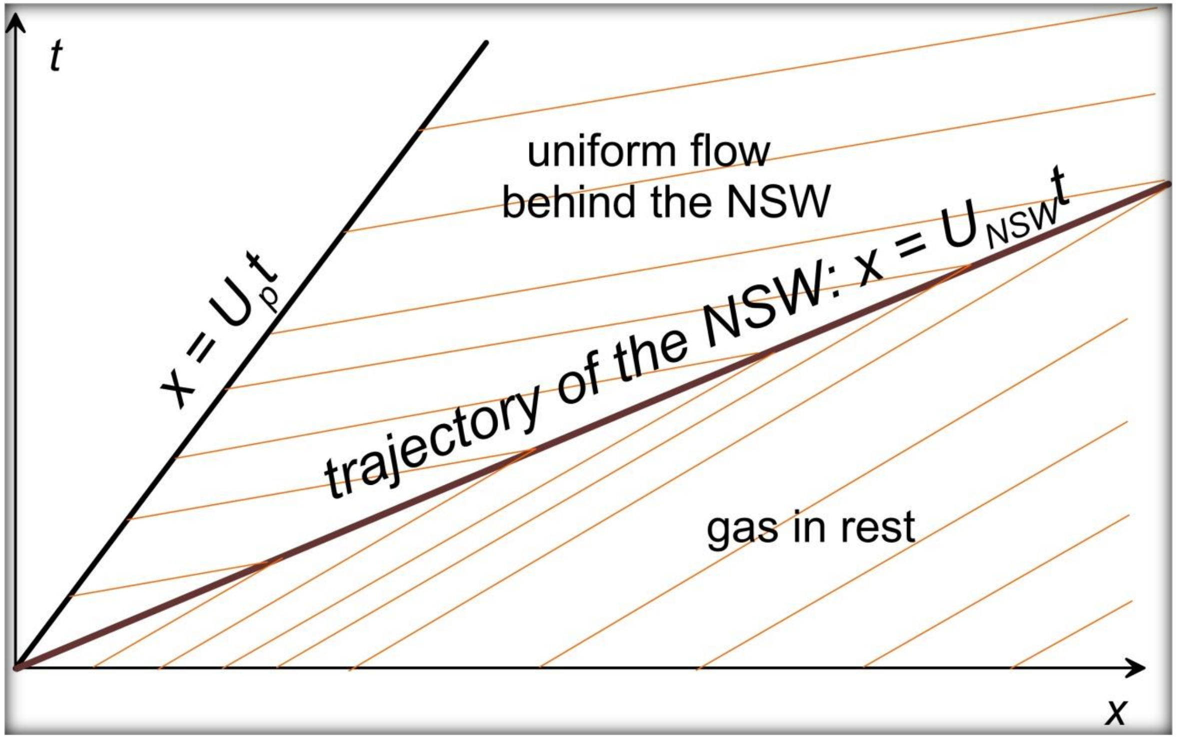
$$\begin{cases} t(T) = T + \dfrac{a_0 + \frac{k-1}{2} X'_P(T)}{\frac{k+1}{2} X''_P(T)} \\ x(T) = X_P(T) + \dfrac{[a_0 + \frac{k+1}{2} X'_P(T)][a_0 + \frac{k-1}{2} X'_P(T)]}{\frac{k+1}{2} X''_P(T)} \end{cases}$$

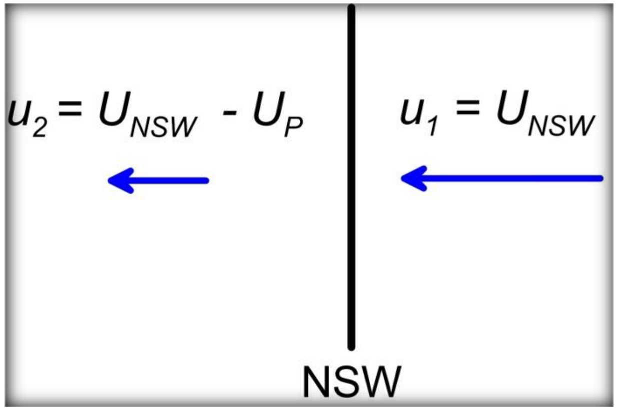
A discontinuity begins to developed at the intersection of envelope line and the front of the compression wave, i.e., the C_+ characteristics $x = a_0 t$. This point corresponds to $T = 0$.
Hence,

$$t_b = \frac{2}{k+1}\frac{a_0}{X''_P(0)} \quad , \quad x_b = a_0 t_b = \frac{2}{k+1}\frac{a_o^2}{X''_P(0)}$$

Note that **the instant** t_b **is determined by the initial acceleration** of the piston!

Consider the case when $X_P''(0) \to \infty$ meaning that the piston attains the finite speed U_P is no time. Assume also that this speed is a finale one, i.e., it keeps steady. According to the above formulae, the shock wave appears immediately – see the figure below.





Determination of U_{NSW}:

$$\begin{cases} U_{NSW}(U_{NSW} - U_P) = a_*^2 \\ \frac{1}{2}U_{NSW}^2 + \frac{1}{k-1}a_0^2 = \frac{k+1}{2(k-1)}a_*^2 \end{cases}$$

Solution: $$U_{NSW} = \tfrac{k+1}{4}U_P + \sqrt{a_0^2 + (\tfrac{k+1}{4}U_P)^2}$$

<u>**Exercise:**</u> Calculate the slope of C_+ characteristics in the region behind the NSW. Any interesting observations?